ON MOTION OF KOWALEWSKI'S GYROSCOPE

(OB ODNOM DVIZHENII GIROSKOPA KOVALEVSKOI)

PMM Vol.28, № 3, 1964, pp. 521-522

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(Received September 20, 1963)

In the paper [1], using the small-parameter method, we found the first terms of the series expansion of the periodic solutions of the equations of motion of a heavy rigid body about a fixed point when the body spins rapidly about the z-axis. The study of the corresponding motion of this body depended essentially on the constant component of the speed of precession which vanishes under the approximation taken, for example, in the case

$$z_0 = 0, \ (A - C)(B - C) / AB = 1/4 \tag{0.1}$$

We shall calculate below the constant component of the speed of precession for Kowalewski's gyroscope in the Bobylev-Steklov case (the solution of the initial system of equations of motion is periodic, and the condition $A = B = 2C$, $y_0 = z_0 = 0$ satisfies equation (0.1)), and we investigate the resulting motion.

1. As is known [2], the equations of motion of a heavy rigid body about a fixed point in Kowalewski's case

$$2dp / dt = qr, \qquad 2dq / dt = -pr - \varepsilon c^2\gamma'', \qquad dr / dt = \varepsilon c^2\gamma', \qquad c^2 = Mg\,|x_0|\,C^{-1} \tag{1.1}$$

$$d\gamma / dt = r\gamma' - q\gamma'', \qquad d\gamma' / dt = p\gamma'' - r\gamma, \qquad d\gamma'' / dt = q\gamma - p\gamma', \qquad \varepsilon = \operatorname{sign} x_0$$

are satisfied by the particular solution

$$p = p_0, \qquad q = 0, \qquad r = -\varepsilon c^2 p_0^{-1}\gamma''$$

and the system (1.1) has, in addition, two first integrals

$$\gamma - {}^1/_2\varepsilon c^2 p_0^{-2}\gamma''^2 = \gamma_0 - {}^1/_2\varepsilon c^2 p_0^{-2}\gamma_0''^2, \qquad \gamma^2 + \gamma'^2 + \gamma''^2 = 1$$

Here p_0, q_0, r_0, γ_0, γ_0' and γ_0'' denote the initial values of the variables.

We shall assume that at the initial instant of time the axis of symmetry of the ellipsoid of inertia z is inclined to the vertical at some angle θ_0 and that a high angular velocity r_0 about this axis is imparted to the rigid body. Without losing generality, we select as the movable coordinate system a system in which at the initial instant of time the positive z and x axes do not form an obtuse angle with the direction of the gravitational force. Then in this coordinate system the quantities r_0 and x_0 may take on either positive or negative values, and the initial values γ_0 and γ_0'' will satisfy conditions $\gamma_0 \geqslant 0$, $\gamma_0'' \geqslant 0$.

Since the system (1.1) is self-contained, we shall assume that $\gamma_0' = 0$.

Under the above assumptions and condition

$$0 < \gamma_0'' < 1 \tag{1.2}$$

(the cases $\gamma_0'' = 0$ and $\gamma_0'' = 1$ are discussed below) the solution of equations (1.1) is found in the form

$$p = -\varepsilon c\mu, \qquad q = 0, \qquad r = \frac{r_0 h_1}{\gamma_0''}\,dn\,u, \qquad \gamma = \sqrt{1-\gamma_0''^2} + \frac{\varepsilon\,(h_1^2 dn^2 u - \gamma_0''^2)}{2\mu^2},$$

$$\gamma' = \frac{k^2 h_1^2}{2\mu^2}\,\mathrm{sn}\,u\,\mathrm{cn}\,u, \qquad \gamma'' = h_1\,\mathrm{du}\,u \tag{1.3}$$

$$u = -\frac{h_1\tau}{2} + u_0, \qquad \tau = \frac{\varepsilon r_0 t}{\gamma_0''}, \qquad k^2 = 1 - \frac{h_2^2}{h_1^2}, \qquad \mu = \frac{c\gamma_0''}{r_0}$$

$$h_j^2 = \gamma_0''^2 - 2\varepsilon\mu^2\sqrt{1-\gamma_0''^2} - 2\mu^4 - (-1)^j\,2\mu^2\,[1 - \gamma_0''^2 + 2\varepsilon\mu^2\sqrt{1-\gamma_0''^2} + \mu^4]$$

$$(j = 1, 2)$$

Here μ is a small parameter. We note the relation

$$k^2 = 4\mu^2\,\frac{\sqrt{1-\gamma_0''^2} + \varepsilon\mu^2}{h_1^2} + \mu^6(\ldots)$$

$$h_1 = \gamma_0''\left[1 - \frac{k^2(\varepsilon-1)}{4}\right] + k^4(\ldots), \qquad u_0 = \frac{\pi(1-\varepsilon)}{4} + k^2(\ldots) \tag{1.4}$$

2. For an analysis of the motion of Kowalewski's gyroscope we introduce the Euler angles θ, φ and ψ

$$\cos\theta = \gamma'', \qquad \frac{d\psi}{dt} = \frac{p\gamma + q\gamma'}{1-\gamma''^2}, \qquad \frac{d\varphi}{dt} = r - \frac{d\psi}{dt}\cos\theta \qquad \left(\mathrm{tg}\,\varphi_0 = \frac{\gamma_0}{\gamma_0'}\right) \tag{2.1}$$

From the first formula of (2.1) and Equations (1.3) and (1.4), we find

$$\cos\theta = \gamma_0''\left[1 + \mu^2\,\frac{\varepsilon\sqrt{1-\gamma_0''^2}}{\gamma_0''^2}(\cos r_0 t - 1)\right] + \mu^4(\ldots) \tag{2.2}$$

The expression for the angle of precession takes on the form [3]

$$\psi - \psi_0 = -\frac{\varepsilon\mu^2\sin r_0 t}{\gamma_0''\sqrt{1-\gamma_0''^2}} + \frac{\mu^3 c N(m_0)}{2\gamma_0''^2}\,t + \ldots \tag{2.3}$$

$$N(m_0) = \frac{m_0 + 1/2}{4m_0(m_0+1)} + \frac{3}{2} + m_0(3 - 2\varepsilon), \qquad m_0 = \frac{\gamma_0''^2}{1-\gamma_0''^2}$$

We note on the basis of (1.2) that $N(m_0) > 0$; from this it follows that for sufficiently small μ, the constant component of the speed of precession will never vanish.

From the last two formulas of (2.1) we have for the spin angle

$$\varphi - \tfrac{1}{2}\pi = \left(r_0 - \frac{Mgx_0}{Cr_0}\sqrt{1-\gamma_0''^2}\right)t + \mu^2(\ldots) \tag{2.4}$$

To the three arbitrary constants $\cos\theta_0 = \gamma_0''$, ψ_0 and r_0 (where r_0 is large), which appear in these formulas, we may add [1] (by replacing t by $t+h$) a fourth arbitrary constant φ_0, which is related to h, on the basis of (2.4), by Formula

$$\varphi_0 = \tfrac{1}{2}\pi + r_0 h + \mu(\ldots)$$

Comparing the resulting approximate expressions for the Euler angles with the analogous expressions of [1], we can convince ourselves that they coincide, except for the constant of the speed of precession, whose order of smallness, μ^3, is higher than that considered in [1]. For this reason, the center of a spherical rectangle in which is inscribed the ellipse which, in a first approximation, is the trajectory of the axis of symmetry of the gyroscope on the fixed unit sphere [1], will move along a corresponding parallel. The speed of displacement will be equal to the constant component of the speed of precession.

The spin of Kowalewski's gyroscope, as is evident from Formula (2.4), will differ only slightly from uniform rotation with a high angular velocity r_0.

3. Let us consider the motion of Kowalewski's gyroscope for the cases $\gamma_0'' = 0$ and $\gamma_0'' = 1$.

If $\gamma_0'' = 0$ $(r_0 \neq 0)$, we obtain Equations $p_0 = 0$ and $\gamma'' = 0$, which leads to the case of physical pendulum.

In order to study the case $\gamma_0'' = 1$, we select the moving coordinate axis x in such a way that $x_0 > 0$. Then, setting

$$\gamma_0 = \gamma_0' = u_0 = 0, \gamma_0'' = \varepsilon = h_1 = 1, \quad k = 2\mu^2, \quad u = {}^1/_2 r_0 t$$

according to the formulas of Sections 1 and 2, retaining only the first terms of the series expansion in powers of μ and omitting the arbitrary constants which are not relevant to what follows, we have

$$\cos\theta = 1 - 2\mu^4 \sin^2 ({}^1/_2 r_0 t) + \ldots, \quad \psi = {}^1/_2 r_0 t, \qquad \varphi = {}^1/_2 r_0 t + \ldots \qquad (3.1)$$

Let us draw a fixed sphere of unit radius about the fixed point and consider on this sphere a circle of radius a $(a = \sqrt{2}\mu^2)$ with its center at the point of intersection of the sphere with the downward direction of the vertical. Then the trajectory of the axis of symmetry of the gyroscope in this case will be the curve $\theta = a \sin\psi$, consisting of two circles of radius $\frac{1}{2}a$ which are tangent to each other. Describing this sinusoidal spiral, the axis of symmetry will, in a first approximation, perform a periodic motion with period $T = 4\pi/r_0$. The characteristic rotation of the body, as is evident from the last formula of (3.1), will differ only slightly from uniform revolution with a high angular velocity $\frac{1}{2}r_0$.

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Translated by A.C.